REMARKS ON RATIONAL SOLUTIONS OF YANG-BAXTER EQUATIONS

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Abstract. In this article, we study unitary rational solutions of the associative Yang-Baxter equation with three spectral parameters. We explain how such solutions arise from the geometry of vector bundles on a cuspidal cubic curve. Moreover, we investigate how these solutions are related to the quantum and classical Yang-Baxter equations.

1. Introduction

In this article we study solutions of the associative Yang-Baxter equation (AYBE)

$$\begin{aligned} &r^{12}(u;y_1,y_2)\,r^{23}(u+v;y_2,y_3) = \\ &= r^{13}(u+v;y_1,y_3)\,r^{12}(-v;y_1,y_2) + r^{23}(v;y_2,y_3)\,r^{13}(u;y_1,y_3). \end{aligned} \tag{1}$$

Here $r : (\mathbb{C}^3, 0) \to A \otimes A$ is the germ of a meromorphic function for $A = \mathrm{Mat}_{n\times n}(\mathbb{C})$. Moreover, for $i \neq j \in \{1, 2, 3\}$, we use the notation $r^{ij} = r \circ \rho_{ij}$ for the composition of r with the canonical embedding $\rho_{ij} : A^{\otimes 2} \to A^{\otimes 3}$, e.g. $\rho_{13}(a \otimes b) = a \otimes \mathbb{1} \otimes b$. A solution r of (1) is called unitary if

$$r^{12}(v; y_1, y_2) = -r^{21}(-v; y_2, y_1)$$

and non-degenerate if the tensor $r(v; y_1, y_2) \in A \otimes A$ is non-degenerate for generic v, y_1, y_2. Non-degenerate unitary solutions of (1) have previously been studied by Polishchuk [3, 4] and Burban, Kreußler [2].

We will focus on solutions r of (1) satisfying the following Ansatz:

$$r(v; y_1, y_2) = \frac{\mathbb{1} \otimes \mathbb{1}}{v} + r_0(y_1, y_2) + v\, r_1(y_1, y_2) + v^2\, r_2(y_1, y_2) + \dots \tag{2}$$

This is motivated by the following fact. Let $\mathrm{pr} : A \to \mathfrak{sl}_n(\mathbb{C})$ denote the canonical projection $X \mapsto X - \frac{\mathrm{tr} X}{n}\mathbb{1}$. It is not difficult to show, see [3, 2], that $\bar{r}_0(y_1, y_2) = (\mathrm{pr} \otimes \mathrm{pr})\,(r_0(y_1, y_2))$ satisfies the classical Yang-Baxter equation (CYBE)

$$\begin{aligned} &\Big[\bar{r}_0^{12}(y_1,y_2), \bar{r}_0^{23}(y_2,y_3)\Big] + \Big[\bar{r}_0^{12}(y_1,y_2), \bar{r}_0^{13}(y_1,y_3)\Big] = \\ &= -\Big[\bar{r}_0^{13}(y_1,y_3), \bar{r}_0^{23}(y_2,y_3)\Big]. \end{aligned} \tag{3}$$

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In section 2 we present an algorithm attaching to any pair of coprime integers (n, d) with $0 < d < n$ a non-degenerate unitary solution $r_{(n,d)}$ of the AYBE (1). The idea behind this algorithm is the computation of certain triple Massey products in the bounded derived category of coherent sheaves $\mathrm{D}^b(\mathrm{Coh}(E))$ for a cuspidal cubic curve $E = V(y^2z - x^3) \subset \mathbb{P}^2$. In all examples computed so far, solutions produced by this algorithm satisfy (2). Moreover, in all these examples $\bar{r}_0$ has no infinitesimal symmetries, i.e. there is no non-trivial $a \in \mathfrak{sl}_n(\mathbb{C})$ such that

$$\Big[\bar{r}_0(y_1, y_2), a \otimes \mathbb{1} + \mathbb{1} \otimes a\Big] = 0.$$

As observed by Polishchuk [4], certain unitary solutions of (1) are closely related with the quantum Yang-Baxter equation. The remaining sections of this paper are dedicated to the generalization of his results. Our main result is the following:

Theorem 1.1. *Let $r(v; y_1, y_2)$ be a non-degenerate unitary solution of the AYBE (1) of the form (2). If $\bar{r}_0(y_1, y_2) = (\mathrm{pr} \otimes \mathrm{pr})\,(r_0(y_1, y_2))$ has no infinitesimal symmetries, then the following hold.*

i) For fixed $v_0 \in \mathbb{C}^\times$, $\tilde{r}(y_1, y_2) = r(v_0; y_1, y_2)$ is a solution of the quantum Yang-Baxter equation (QYBE)

$$\tilde{r}^{12}(y_1, y_2)\,\tilde{r}^{13}(y_1, y_3)\,\tilde{r}^{23}(y_2, y_3) = \tilde{r}^{23}(y_2, y_3)\,\tilde{r}^{13}(y_1, y_3)\,\tilde{r}^{12}(y_1, y_2). \tag{4}$$

ii) Let $s(v; y_1, y_2)$ be another non-degenerate unitary solution of the AYBE (1) of the form (2) with $(\mathrm{pr} \otimes \mathrm{pr})\,(s_0) = \bar{r}_0$. Then there exists a meromorphic function $g : \mathbb{C} \to \mathbb{C}$ such that

$$s(v; y_1, y_2) = \exp\Big(v\,(g(y_2) - g(y_1))\Big) r(v; y_1, y_2).$$

If additionally both r and s can be obtained by the procedure described in section 2, then g is holomorphic. Thus the above equation states that r and s are gauge equivalent in that case.

This result is proved in two steps, which are theorem 5.1 respectively theorem 6.1 and corollary 6.4. Let us add the following remarks:

• Theorem 1.1 was shown by Polishchuk, see [4, Theorem 1.4] and [3, Theorem 6], in the case when $r(v; y_1, y_2)$ depends only on v and the difference $y = y_1 - y_2$. However, solutions obtained by the algorithm presented in section 2 do not have this property.
• The proof of theorem 1.1 is purely analytical.
• The solutions $\bar{r}_0$ of the CYBE (3) obtained by the procedure presented in section 2 belong to the class of *rational* solutions. Rational solutions of the CYBE (3) have been classified by Stolin [5]. As we shall show in a subsequent paper [1], the solutions $\bar{r}_0$ produced by our algorithm form a subclass of rational solutions which can be intrinsically described in terms of Stolin's classification. One of the benefits

of our method is that theorem 1.1 gives an explicit way of lifting these solutions of the CYBE to solutions of the QYBE.
• There are solutions r of (1) which satisfy (4) even though $\bar{r}_0$ has infinitesimal symmetries. An example is given by the following function, depending only on the difference $y = y_1 - y_2$:

$$r(v, y) = \frac{1}{2v}\mathbb{1} \otimes \mathbb{1} + \frac{1}{y}\left(e_{11} \otimes e_{11} + e_{22} \otimes e_{22} + e_{12} \otimes e_{21} + e_{21} \otimes e_{12}\right).$$

The corresponding solution of the CYBE (3) is the rational solution of Yang

$$\bar{r}_0(v) = \frac{1}{y}\left(\frac{1}{2}h \otimes h + e_{12} \otimes e_{21} + e_{21} \otimes e_{12}\right) \in \mathfrak{sl}_2(\mathbb{C}) \otimes \mathfrak{sl}_2(\mathbb{C})$$

whose infinitesimal symmetries are given by all non-zero elements of $\mathfrak{sl}_2(\mathbb{C})$. Hence, there might exist a generalization of theorem 1.1 i).

2. The algorithm

In this section we present an algorithm which takes as input a pair of coprime numbers (n, d) with $0 < d < n$ and produces a non-degenerate unitary solution $r_{(n,d)}$ of the AYBE (1) with values in $A \otimes A$ where $A = \mathrm{Mat}_{n\times n}(\mathbb{C})$. The algorithm is due to Burban and Kreußler [2, Section 10], see in particular [2, Algorithm 10.7]. The actual content of the described procedure is the computation of certain triple Massey products in the bounded derived category of coherent sheaves $\mathrm{D}^b(\mathrm{Coh}(E))$ for a cuspidal cubic curve $E = V(y^2z - x^3) \subset \mathbb{P}^2$.

⋆ **Step 1: construction of the matrix $J = J(n-d, d)$.**
We introduce the following map defined on all tuples of coprime integers $(a, b) \neq (1, 1)$:

$$\epsilon(a, b) = \begin{cases} (a-b, b), & a > b \\ (a, b-a), & a < b \end{cases}$$

By assumption (n, d) is a tuple of coprime integers. Hence it induces a finite sequence of tuples ending with $(1, 1)$, defined as follows. We put $(a_0, b_0) = (n - d, d)$ and, as long as $(a_i, b_i) \neq (1, 1)$, we set $(a_{i+1}, b_{i+1}) = \epsilon(a_i, b_i)$. Next, let

$$J(1, 1) = \left(\begin{array}{c|c} 0 & 1 \\ \hline 0 & 0 \end{array}\right) \in \mathrm{Mat}_{2\times 2}(\mathbb{C}).$$

Assuming

$$J(a, b) = \left(\begin{array}{c|c} J_1 & J_2 \\ \hline 0 & J_3 \end{array}\right)$$

with $J_1 \in \mathrm{Mat}_{a\times a}(\mathbb{C})$ and $J_3 \in \mathrm{Mat}_{b\times b}(\mathbb{C})$ has already been defined and that $(a,b) = \epsilon(p,q)$, we set

$$J(p,q) = \begin{cases} \left(\begin{array}{c|cc} 0 & \mathbb{1} & 0 \\ \hline 0 & J_1 & J_2 \\ 0 & 0 & J_3 \end{array}\right), & p = a \\[2em] \left(\begin{array}{cc|c} J_1 & J_2 & 0 \\ 0 & J_3 & \mathbb{1} \\ \hline 0 & 0 & 0 \end{array}\right), & q = b. \end{cases}$$

Hence, to (n,d) we may associate the $n \times n$ matrix $J = J(n-d,d)$ that is obtained from the matrix $J(1,1)$ and the sequence $\big\{(n-d,d),...,(1,1)\big\}$ by applying the recursive procedure described above.

Example 2.1. Let $(n,d) = (5,2)$. The induced sequence is $\big\{(3,2),(1,2),(1,1)\big\}$ and $J = J(3,2)$ is constructed as follows

$$\left(\begin{array}{c|c} 0 & 1 \\ \hline 0 & 0 \end{array}\right) \to \left(\begin{array}{c|cc} 0 & 1 & 0 \\ \hline 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right) \to \left(\begin{array}{ccc|cc} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

$\star$ **Step 2: definition of $\mathbf{Sol}_{n,d}^{v,y_1}$, $\mathbf{res}_{y_1}$ and $\mathbf{ev}_{y_2}$.**
For the partition of $A = \mathrm{Mat}_{n\times n}(\mathbb{C})$ induced by the partition of J as in step 1, we introduce the following subspace of the polynomial ring $A[z]$:

$$W_{n,d} = \left\{ F(z) = \left(\begin{array}{c|c} W & X \\ \hline Y & Z \end{array}\right) + \left(\begin{array}{c|c} W' & 0 \\ \hline Y' & Z' \end{array}\right) z + \left(\begin{array}{c|c} 0 & 0 \\ \hline Y'' & 0 \end{array}\right) z^2 \right\}.$$

Next, for $F(z) \in W_{n,d}$ we denote

$$F_0 = \left(\begin{array}{c|c} W' & X \\ \hline Y'' & Z' \end{array}\right) \text{ and } F_\epsilon = \left(\begin{array}{c|c} W & 0 \\ \hline Y' & Z \end{array}\right).$$

Then for $v, y_1 \in \mathbb{C}$, we define the following subspace of $W_{n,d}$:

$$\mathrm{Sol}_{n,d}^{v,y_1} = \left\{ F(z) \in W_{n,d} \,\middle|\, [F_0, J] + (y_1 - v)F_0 + F_\epsilon = 0 \right\}.$$

Proposition 2.2. [2, Section 10] *The vector space* $\mathrm{Sol}_{n,d}^{v,y_1}$ *has dimension* n^2 *and for* $y_1 \neq y_2 \in \mathbb{C}$, $\mathrm{ev}_{y_2} : \mathrm{Sol}_{n,d}^{v,y_1} \to A$ *defined by* $\mathrm{ev}_{y_2}\big(F(z)\big) = \frac{1}{y_2 - y_1} F(y_2)$ *is an isomorphism. For* $v \neq 0$, $\mathrm{res}_{y_1} : \mathrm{Sol}_{n,d}^{v,y_1} \to A$ *given by* $\mathrm{res}_{y_1}\big(F(z)\big) = F(y_1)$ *is an isomorphism as well.*

We will assume $v \neq 0$ and $y_1 \neq y_2$ for the rest of this section. Then we get a linear automorphism $\tilde{r}_{(n,d)}$ of the matrix algebra A given by the formula

$\tilde{r}_{(n,d)}(v;y_1,y_2) = \mathrm{ev}_{y_2} \circ \mathrm{res}_{y_1}^{-1}$.

$\star$ **Step 3: definition of the tensor $r_{(n,d)}(v;y_1,y_2)$.**
Note that we have a canonical isomorphism of vector spaces

$$\mathrm{can} : A \otimes A \to \mathrm{End}_{\mathbb{C}}(A),\ X \otimes Y \mapsto \big(Z \mapsto \mathrm{tr}(XZ)Y\big).$$

For fixed v, y_1, y_2, we set $r_{(n,d)}(v;y_1,y_2) = \mathrm{can}^{-1}\big(\tilde{r}_{(n,d)}(v;y_1,y_2)\big)$. The proof of the following theorem is contained in [2, Section 10].

Theorem 2.3. *The tensor-valued function $r_{(n,d)} : \Big(\mathbb{C}^3_{(v;y_1,y_2)}, 0\Big) \to A \otimes A$ is a non-degenerate unitary solution of (1). Moreover $r_{(n,d)}(v;y_1,y_2)$ is holomorphic on $\big(\mathbb{C}^3 \setminus V\big(v(y_1-y_2)\big)\big)$.*

Example 2.4. For any $n \in \mathbb{N}$, let $P = \sum_{1 \le i,j \le n} e_{ij} \otimes e_{ji} \in A \otimes A$.

i) Let $(n,d) = (2,1)$. Then we have

$$r_{(2,1)}(v;y_1,y_2) = \frac{1}{2v}\mathbb{1} \otimes \mathbb{1} + \frac{1}{y_2 - y_1}P +$$
$$+ (v-y_1)\, e_{21} \otimes \check{h} + (v+y_2)\, \check{h} \otimes e_{21} - \frac{v(v-y_1)(v+y_2)}{2} e_{21} \otimes e_{21},$$

where $\check{h} = \mathrm{diag}(\frac{1}{2}, -\frac{1}{2})$.

ii) Let $(n,d) = (3,1)$. Then we have

$$r_{(3,1)}(v;y_1,y_2) = \frac{1}{3v}\mathbb{1} \otimes \mathbb{1} + \frac{1}{y_2 - y_1}P -$$
$$-e_{21} \otimes \check{h}_1 + \check{h}_1 \otimes e_{21} + e_{32} \otimes e_{12} - e_{12} \otimes e_{32} - y_1 e_{32} \otimes \check{h}_2 + y_2 \check{h}_2 \otimes e_{32} +$$
$$+(v-y_1)e_{31} \otimes e_{12} + (v+y_2)e_{12} \otimes e_{31} + v e_{32} \otimes (e_{11} - e_{33}) + v(e_{11} - e_{33}) \otimes e_{32} +$$
$$+\frac{1}{3}v(y_1 - 3v)e_{32} \otimes e_{21} + \frac{1}{3}v(y_2 + 3v)e_{21} \otimes e_{32} + v(v-y_1)e_{31} \otimes \check{h}_1 -$$
$$-v(v+y_2)\check{h}_1 \otimes e_{31} + \frac{2}{3}v^2(y_1 - v)e_{31} \otimes e_{21} - \frac{2}{3}v^2(y_2+v)e_{21} \otimes e_{31} +$$
$$+\frac{1}{3}v^2(y_2+v)(3v-y_1)e_{32} \otimes e_{31} + \frac{1}{3}v^2(y_1+v)(3v+y_2)e_{31} \otimes e_{32} +$$
$$+\frac{2}{3}v e_{21} \otimes e_{21} + \frac{2}{3}v^3(v-y_1)(v+y_2)e_{31} \otimes e_{31}$$
$$+\frac{1}{3}v(-6v^2 + 3v(y_1 - y_2) + 2y_1y_2)e_{32} \otimes e_{32},$$

where $\check{h}_1 = \mathrm{diag}(\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3})$ and $\check{h}_2 = \mathrm{diag}(\frac{1}{3}, \frac{1}{3}, -\frac{2}{3})$.

Next, recall [2, Lemma 2.11], which is based upon [3, Lemma 1.2]:

Lemma 2.5. *Let $r(v; y_1, y_2)$ be a unitary solution of (1) of the form (2). Then $\bar{r}_0(y_1, y_2) = (\mathrm{pr} \otimes \mathrm{pr})\,(r_0(y_1, y_2))$ is a unitary solution of the CYBE (3).*

Combining this result with example (2.4), we derive that $r_{(2,1)}$ and $r_{(3,1)}$ induce solutions of of the CYBE (3). Let us denote these by $c_{(2,1)}$ and $c_{(3,1)}$ respectively. Also, let Ω be the Casimir element of $\mathfrak{sl}_n(\mathbb{C}) \otimes \mathfrak{sl}_n(\mathbb{C})$ with respect to the trace form $(x, y) \mapsto \mathrm{tr}(x \cdot y)$

$$\Omega = \sum_{1 \le i \ne j \le n} e_{i,j} \otimes e_{j,i} + \sum_{1 \le l \le n-1} h_l \otimes \check{h}_l.$$

Observing that $(\mathrm{pr} \otimes \mathrm{pr})\,(P) = \Omega$, we derive the following formulae:

$$c_{(2,1)}(y_1, y_2) = \frac{\Omega}{y_2 - y_1} + y_2 \check{h} \otimes e_{21} - y_1 e_{21} \otimes \check{h} \in \mathfrak{sl}_2(\mathbb{C}) \otimes \mathfrak{sl}_2(\mathbb{C})$$

and

$$\begin{aligned} c_{(3,1)}(y_1, y_2) = \frac{\Omega}{y_2 - y_1} &+ y_2 \check{h}_2 \otimes e_{32} - y_1 e_{32} \otimes \check{h}_2 + y_2 e_{12} \otimes e_{31} - y_1 e_{31} \otimes e_{12} - \\ &- e_{21} \otimes \check{h}_1 + \check{h}_1 \otimes e_{21} + e_{32} \otimes e_{12} - e_{12} \otimes e_{32} \in \mathfrak{sl}_3(\mathbb{C}) \otimes \mathfrak{sl}_3(\mathbb{C}). \end{aligned}$$

It can be verified that neither $c_{(2,1)}$ nor $c_{(3,1)}$ has any infinitesimal symmetries. Thus theorem 1.1 1) yields that for fixed $v_0 \in \mathbb{C}^\times$, both $r_{(2,1)}(v_0; y_1, y_2)$ and $r_{(3,1)}(v_0; y_1, y_2)$ satisfy the QYBE (4).

3. Gauge equivalence and general results on the AYBE

In this section we explain the notion of gauge equivalence and collect some useful results for solutions of (1).

In order to deal with gauge equivalences in a correct way, we have to consider a more general form of the AYBE in four variables

$$\begin{aligned} & r^{12}(v_1, v_2; y_1, y_2)\, r^{23}(v_1, v_3; y_2, y_3) = \\ &= r^{13}(v_1, v_3; y_1, y_3)\, r^{12}(v_3, v_2; y_1, y_2) + r^{23}(v_2, v_3; y_2, y_3)\, r^{13}(v_1, v_2; y_1, y_3) \end{aligned} \tag{5}$$

where r now denotes the germ of a meromorphic function $r : (\mathbb{C}^4, 0) \to A \otimes A$. From the point of view of algebraic geometry, (5) is more natural than (1), see [2]. Note that solutions of (1) are just solutions of (5) depending on the difference of the first pair of spectral parameters $r(v_1, v_2; y_1, y_2) = r(v_1 - v_2; y_1, y_2) = r(v; y_1, y_2)$. As to the definition of gauge equivalence, this is given as follows:

Definition 3.1. Let $\phi : (\mathbb{C}^2, 0) \to \mathrm{GL}_n(\mathbb{C})$ be the germ of a holomorphic function and let $r(v_1, v_2; y_1, y_2)$ be a solution of (5). Then the tensor valued function

$$r'(v_1, v_2; y_1, y_2) = \Big(\phi(v_1; y_1) \otimes \phi(v_2; y_2)\Big)\, r(v_1, v_2; y_1, y_2)\, \Big(\phi^{-1}(v_2; y_1) \otimes \phi^{-1}(v_1; y_2)\Big)$$

is also a solution of (5). The solutions r and r' are said to be gauge equivalent and ϕ is called a gauge transformation.

Example 3.2. Let $r(v_1, v_2; y_1, y_2) \in A \otimes A$ be a solution of (5), $c \in \mathbb{C}$ and $\phi(v, y) = \exp(cvy) \cdot \mathbb{1} : (\mathbb{C}^2, 0) \to \mathrm{GL}_n(\mathbb{C})$ be a gauge transformation. Then

$$\exp\Big(c\,(v_2 - v_1)\,(y_2 - y_1)\Big)\, r(v_1, v_2; y_1, y_2)$$

is a solution of (5), gauge equivalent to r.

Similarly, assume that $r(v_1, v_2; y_1, y_2) = r(v; y_1, y_2)$ is a solution of (5) which depends only on $v = v_1 - v_2, y_1, y_2$. Thus $r(v; y_1, y_2)$ is a solution of (1). Consider the gauge transformation $\phi(v, y) = \exp(vg(y)) \cdot \mathbb{1} : (\mathbb{C}^2, 0) \to \mathrm{GL}_n(\mathbb{C})$ for some holomorphic function $g : \mathbb{C} \to \mathbb{C}$. Then $r'(v; y_1, y_2) = \exp\big(v\,(g(y_2) - g(y_1))\big)\, r(v; y_1, y_2)$ is a solution of (1) as well.

In the remainder of this section, we list some basic results on solutions of the AYBE which we shall need in the next sections.

Lemma 3.3. [2, Lemma 2.7] *Let $r(v_1, v_2; y_1, y_2)$ be a unitary solution of (5). Then writing $r^{ij}(v_1, v_2)$ as short-hand for $r^{ij}(v_1, v_2; y_i, y_j)$, r also satisfies the "dual equation"*

$$r^{23}(v_2, v_3)\, r^{12}(v_1, v_3) = r^{12}(v_1, v_2)\, r^{13}(v_2, v_3) + r^{13}(v_1, v_3)\, r^{23}(v_2, v_1).$$

Corollary 3.4. *If $r(v; y_1, y_2)$ is a unitary solution of (1), then we also have*

$$\begin{aligned} & r^{23}(u+v; y_2, y_3)\, r^{12}(u; y_1, y_2) = \\ & = r^{12}(-v; y_1, y_2)\, r^{13}(u+v; y_1, y_3) + r^{13}(u; y_1, y_3)\, r^{23}(v; y_2, y_3). \end{aligned} \tag{6}$$

The proof of the next lemma is essentially contained in the proof of [3, Theorem 5].

Lemma 3.5. *Let $r(v; y_1, y_2)$ be a unitary solution of (1) of the form (2). Then r is uniquely determined by r_0 and r_1. Moreover, we have*

$$\begin{aligned} & r_1^{12}(y_1, y_2) + r_1^{13}(y_1, y_3) + r_1^{23}(y_2, y_3) = \\ & = r_0^{12}(y_1, y_2)\, r_0^{13}(y_1, y_3) - r_0^{23}(y_2, y_3)\, r_0^{12}(y_1, y_2) + r_0^{13}(y_1, y_3)\, r_0^{23}(y_2, y_3). \end{aligned} \tag{7}$$

Proof. First we show that r is uniquely determined by r_0, r_1 and r_2. To this end, we fix $k > 2$ and show how to construct r_k from $\{r_i\}_{0 \le i \le k-1}$. Let us insert the Laurent expansion (2) of r into (6) and examine the terms of total degree $k - 1$ in the variables u and v. We derive the equation

$$\begin{aligned} & r_k^{12}(y_1, y_2)\, \left[\frac{(-v)^k}{u+v} - \frac{u^k}{u+v}\right] + r_k^{13}(y_1, y_3)\, \left[\frac{u^k}{v} - \frac{(u+v)^k}{v}\right] + \\ & + r_k^{23}(y_2, y_3)\, \left[\frac{v^k}{u} - \frac{(u+v)^k}{u}\right] = \ldots \end{aligned} \tag{8}$$

where the ride-hand side contains terms r_i with $i < k$ only. The polynomials in u and v on the left-hand side are linearly independent for $k > 2$. Indeed, if we place

everything over a common denominator and focus on the coefficients of u^k in the respective terms

$$u(-v)^{k+1} - vu^{k+1},\ u^{k+1}(u+v) - u(u+v)^{k+1},\ (u+v)v^{k+1} - v(u+v)^{k+1}$$

then these are $-vu$, $u(u+v) - \binom{k+1}{2}v^2$ and $-(k+1)\,v^2$ respectively. This proves our claim that r is determined by the r_k with $k \leq 2$.

In the next step, we show that r_2 is already determined by r_0 and r_1. Indeed, for $k = 2$ equation (8) reads

$$(v-u)\ r_2^{12}(y_1, y_2)\ - (2u+v)\ r_2^{13}(y_1, y_3) - (u+2v)\, r_2^{23}(y_2, y_3) = \dots$$

that is

$$-u \cdot \left(r_2^{12}(y_1, y_2) + 2r_2^{13}(y_1, y_3) + r_2^{23}(y_2, y_3)\right) + \\ +v \cdot \left(r_2^{12}(y_1, y_2) - r_2^{13}(y_1, y_3) - 2r_2^{23}(y_2, y_3)\right) = \dots$$

with the the right-hand side depending on r_0 and r_1 only. Let us denote the coefficient of $-u$ on the left-hand side by a, that of v by b. Since a, b are determined by r_0 and r_1 only, so is $\frac{a+2b}{3} = r_2^{12}(y_1, y_2) - r_2^{23}(y_2, y_3)$. Thus, $r_2(y_1, y_2)$ is determined by r_0 and r_1. Putting $k = 1$ in (8), we obtain (7). □

4. Poles of solutions of the AYBE

In this section, we study the poles of solutions of (1) along $y_1 = y_2$. We start with the following easy fact on $P = \sum_{1 \leq i,j \leq n} e_{i,j} \otimes e_{j,i} \in A \otimes A$.

Fact 4.1. *Any tensor $\theta \in A \otimes A$ such that $\theta(x \otimes 1) = (1 \otimes x)\theta$ for all $x \in A$ is a scalar multiple of P. Moreover $P(1 \otimes x) = (x \otimes 1)P$ for any $x \in A$.*

Lemma 4.2. [4, Lemma 1.3] *Let $r(u; y_1, y_2)$ be a non-degenerate unitary solution of (1). Assume that $r(u; y_1, y_2)$ has a pole along $y_1 = y_2$. Then this pole is simple and $\lim_{y_2 \to y_1}(y_1 - y_2)\, r(u; y_1, y_2) = c \cdot P$ for some $c \in \mathbb{C}$.*

Proof. Write $r(u; y_1, y_2) = \alpha(u; y_1 - y_2) + \beta(u; y_1, y_2)$ and assume that no summand of $\beta(u; y_1, y_2)$ depends only on u and $y = y_1 - y_2$. Let $\alpha(u; y) = \frac{\theta(u)}{y^k} + \frac{\eta(u)}{y^{k-1}} + \dots$ be the Laurent expansion near $y = 0$. In order to see that $k \leq 1$, we consider the polar parts in (1) as $y_3 \to y_1$, which yields

$$\theta^{13}(u+v)\, r^{12}(-v; y_1, y_2) + r^{23}(v; y_2, y_1)\, \theta^{13}(u) = 0. \tag{9}$$

Analogously, for $y_2 \to y_1$

$$\theta^{12}(u)\, r^{23}(u+v; y_1, y_3) - r^{13}(u+v; y_1, y_3)\, \theta^{12}(-v) = 0. \tag{10}$$

Let $V \subseteq A$ be the minimal subspace such that $\theta(u) \in V \otimes A$ for all u where $\theta(u)$ is defined. Obviously $r^{23}(u; y_1, y_2)\, \theta^{13}(u) \in V \otimes A \otimes A$ hence by (9) $\theta^{13}(u + v)\, r^{12}(-v; y_1, y_2) \in V \otimes A \otimes A$ as well. Thus, $r^{12}(u; y_1, y_2) \in A_1 \otimes A$, where

$$A_1 = \{a \in A |\, \theta(u)\, (a \otimes 1) \in V \otimes A \,\text{for all } u\}\,.$$

By non-degeneracy $A_1 = A$, thus $VA \subseteq V$. Similarly, using (10), we get $AV \subseteq V$, so that V is a two-sided non-zero ideal in A. Hence $V = A$.

Let us come back to (1). We want to have a look at the coefficient of $(y_1 - y_2)^{1-k}$ in the expansion of (1) near $y_2 - y_1 = h$ equal to zero. The terms contributing to this only depend on $r^{12}(u; y_1, y_1 + h)\, r^{23}(u + v; y_1 + h, y_3)$ and $r^{13}(u + v; y_1, y_3)\, r^{12}(-v; y_1, y_1 + h)$. Thus the coefficient of h^{1-k} consists of two summands, the first one being $\eta^{12}(u)\, r^{23}(u+v; y_1, y_3) - r^{13}(u+v; y_1, y_3)\, \eta^{12}(-v)$ and second one being $\theta^{12}(u)$ times the coefficient of h in $r^{23}(u + v; y_1 + h, y_3)$. Note that for this last summand to be non-zero we must assume $k > 1$. Now $r^{23}(u + v; y_1 + h, y_3) - r^{23}(u + v; y_1, y_3)$ equals the summand of $r^{23}(u + v; y_1 + h, y_3)$ divisible by h, hence the coefficient of h^{1-k} is exactly

$$\eta^{12}(u)\, r^{23}(u + v; y_1, y_3) - r^{13}(u + v; y_1, y_3)\, \eta^{12}(-v) + \theta^{12}(u)\, \frac{\partial r^{23}}{\partial y_1}(u + v; y_1, y_3).$$

Examining the polar parts in the above expression for $y_1 - y_3$ in a neighborhood of zero, we deduce that $\theta^{12}(u)\, \theta^{23}(u + v) = 0$. Setting $v = 0$ this amounts to saying that $\theta(u) = \{\, a \otimes b |\, ab = 0\}$. Since $V = A$ this is a contradiction. Therefore $k = 1$.

Next, we have a look at the polar parts in (6) near $y_3 = y_2$. We deduce $\theta^{23}(u + v)\, r^{12}(u; y_1, y_2) = r^{13}(u; y_1, y_2)\, \theta^{23}(v)$. Hence $r(u; y_1, y_2) \in A \otimes A(u)$, where

$$A(u) = \{a \in A\, |\theta(u + v)(x \otimes 1) = (1 \otimes x)\theta(v) \text{ for all}\, v\, \}\,.$$

Since A is non-degenerate this implies $A(u) = A$ for generic u, in which case $\mathbb{1} \in A(u)$ and thus $\theta(u+v) = \theta(u)$. Hence $\theta = \theta(0)$ is constant. Recalling fact 4.1 finishes the proof. □

Corollary 4.3. [4, Lemma 1.5] *Let $r(u; y_1, y_2)$ be a non-degenerate unitary solution of (1) of the form (2). Then $r(u; y_1, y_2)$ has a simple pole along $y_1 = y_2$ with residue a scalar multiple of P.*

Proof. This is essentially the same proof as that of lemma 1.5 in [4], using lemma 4.2 where Polishchuk refers to lemma 1.3 of his paper. □

5. Quantization of solutions of CYBE coming from solutions of AYBE

In this section we prove part i) of theorem 1.1:

Theorem 5.1. [4, Theorem 1.4] *Let $r(u; y_1, y_2)$ be a non-degenerate unitary solution of (1) of the form (2) and let $\overline{r}_0(y_1, y_2) = (\mathrm{pr} \otimes \mathrm{pr})\, (r_0(y_1, y_2))$.*

(1) *$\overline{r}_0(y_1, y_2)$ is a non-degenerate unitary solution of the CYBE (3).*
(2) *The following conditions are equivalent:*
    (a) *for fixed $u \in \mathbb{C}^\times$, $r(u; y_1, y_2)$ satisfies the QYBE (4).*

(b) *there exits a scalar function* $\varphi(u; y_1, y_2)$ *such that*

$$r(u; y_1, y_2)\, r(-u; y_1, y_2) = \varphi(u; y_1, y_2)\, (\mathbb{1} \otimes \mathbb{1})\,.$$

(c) *for* $i \in \{1, 2\}$ *there exists a scalar function* $\psi_i(y_1, y_2)$ *such that*

$$\frac{\partial}{\partial y_i}\, (r_0(y_1, y_2) - \overline{r}_0(y_1, y_2)) = \psi_i(y_1, y_2)\, (\mathbb{1} \otimes \mathbb{1})\,.$$

(d) *we have*

$$(\mathrm{pr} \otimes \mathrm{pr} \otimes \mathrm{pr}) \; \big[\overline{r}_0^{12}(y_1, y_2)\, \overline{r}_0^{13}(y_1, y_3) -$$
$$- \overline{r}_0^{23}(y_2, y_3)\, \overline{r}_0^{12}(y_1, y_2) + \overline{r}_0^{13}(y_1, y_3)\, \overline{r}_0^{23}(y_2, y_3)\big] = 0.$$

(3) *These conditions are satisfied if* $\overline{r}_0(y_1, y_2)$ *has no infinitesimal symmetries.*

Before proving this statement, we first need to establish some auxiliary results. The reader might wish to postpone checking them and to go to the proof of theorem 5.1 at the end of this section immediately.

Lemma 5.2. [4, Lemma 1.6] *For any triple of variables* u_1, u_2, u_3 *set* $u_{ij} = u_i - u_j$. *Let* $r(u; y_1, y_2)$ *be any unitary solution of (1) and* $s(u; y_1, y_2) = r(u; y_1, y_2)\, r(-u; y_1, y_2)$. *Then*

$$r^{12}(u_{12}; y_1, y_2)\, r^{13}(u_{23}; y_1, y_3)\, r^{23}(u_{12}; y_2, y_3) -$$
$$- r^{23}(u_{23}; y_2, y_3)\, r^{13}(u_{12}; y_1, y_3)\, r^{12}(u_{23}; y_1, y_2) =$$
$$= s^{23}(u_{23}; y_2, y_3)\, r^{13}(u_{13}; y_1, y_3) - r^{13}(u_{13}; y_1, y_3)\, s^{23}(u_{21}; y_2, y_3) =$$
$$= r^{13}(u_{13}; y_1, y_3)\, s^{12}(u_{32}; y_1, y_2) - s^{12}(u_{12}; y_1, y_2)\, r^{13}(u_{13}; y_1, y_3).$$

Proof. Let us write $r^{ij}(u)$ as short-hand for $r^{ij}(u; y_i, y_j)$. Since we may assume $u = u_{12}$, $v = u_{23}$ and $u + v = u_{13}$, (1) may be written as

$$r^{12}(u_{12})\, r^{23}(u_{13}) = r^{13}(u_{13})\, r^{12}(u_{32}) + r^{23}(u_{23})\, r^{13}(u_{12}). \tag{11}$$

Analogously, putting $u = u_{13}$ and $v = u_{21}$, (6) reads

$$r^{23}(u_{23})\, r^{12}(u_{13}) = r^{12}(u_{12})\, r^{13}(u_{23}) + r^{13}(u_{13})\, r^{23}(u_{21}). \tag{12}$$

Multiplying (12) with $r^{23}(u_{12})$ from the right yields

$$r^{23}(u_{23})\, r^{12}(u_{13})\, r^{23}(u_{12}) = r^{12}(u_{12})\, r^{13}(u_{23})\, r^{23}(u_{12}) + r^{13}(u_{13})\, s^{23}(u_{21})$$

while switching u_2 and u_3 in (11) followed by multiplication with $r^{23}(u_{23})$ from the left yields

$$r^{23}(u_{23})\, r^{12}(u_{13})\, r^{23}(u_{12}) = r^{23}(u_{23})\, r^{13}(u_{12})\, r^{12}(u_{23}) + s^{23}(u_{23})\, r^{13}(u_{13}).$$

Subtracting these equations, we end up with

$$r^{12}(u_{12})\, r^{13}(u_{23})\, r^{23}(u_{12}) - r^{23}(u_{23})\, r^{13}(u_{12})\, r^{12}(u_{23}) =$$
$$= s^{23}(u_{23})\, r^{13}(u_{13}) - r^{13}(u_{13})\, s^{23}(u_{21}).$$

Switching indices 1 and 3 and using unitarity of r yields the other identity. $\square$

For the next statement we need the notion of an infinitesimal symmetry of a solution r of (1), which is simply that of an element $a \in \mathfrak{sl}_n(\mathbb{C})$ such that $[r(u; y_1, y_2), a^1 + a^2] = 0$, where $a^1 = a \otimes \mathbb{1}$ and $a^2 = \mathbb{1} \otimes a$.

Lemma 5.3. [4, Lemma 1.7] *Let $r(u; y_1, y_2)$ be a unitary solution of (1) of the form (2) and $s(u; y_1, y_2) = r(u; y_1, y_2)\, r(-u; y_1, y_2)$. Assuming that $r(u; y_1, y_2)$ has a simple pole along $y_1 = y_2$ with residue cP for some $c \in \mathbb{C}$, we have*

$$s(u; y_1, y_2) = a \otimes 1 + 1 \otimes a + (f(u) + g(y_1, y_2))\, \mathbb{1} \otimes \mathbb{1}$$

where $f(u) = f(-u)$, $g(y_1, y_2) = g(y_2, y_1)$ and $a \in \mathfrak{sl}_n(\mathbb{C})$ is an infinitesimal symmetry of $r(u; y_1, y_2)$. Moreover, we may write

$$r_0(y_1, y_2) = \overline{r}_0(y_1, y_2) + \alpha(y_2) \otimes \mathbb{1} - \mathbb{1} \otimes \alpha(y_1) + h(y_1, y_2) \mathbb{1} \otimes \mathbb{1}$$

with $\overline{r}_0(y_1, y_2)$ mapping to $\mathfrak{sl}_n(\mathbb{C}) \otimes \mathfrak{sl}_n(\mathbb{C})$, $\alpha(y)$ to $\mathfrak{sl}_n(\mathbb{C})$, $h(y_1, y_2)$ a scalar function and

$$\alpha(y) = \alpha(0) + \frac{y}{cn} a.$$

Proof. By assumption $r(u; y_1, y_2) = \frac{c}{y_1 - y_2} P + \tilde{r}(u; y_1, y_2)$ where $\tilde{r}(u; y_1, y_2)$ does not have a pole along $y_1 = y_2$. Let us write $r(u; y_{ij})$ and $\tilde{r}(u; y_{ij})$ as short-hand for $r(u; y_i, y_j)$ and $\tilde{r}(u; y_i, y_j)$ respectively. Then starting from (6) we derive that for $v = -u + h$:

(13)
$$\begin{aligned} r^{13}(u; y_{13})\, r^{23}(-u + h; y_{23}) &= r^{23}(h; y_{23})\, r^{12}(u; y_{12}) - r^{12}(u - h; y_{12})\, r^{13}(h; y_{13}) = \\ &= [r^{23}(h; y_{23})\, r^{12}(u; y_{12}) - r^{12}(u; y_{12})\, r^{13}(h; y_{13})] + \\ &\quad + [r^{12}(u; y_{12}) - r^{12}(u - h; y_{12})]\; r^{13}(h; y_{13}). \end{aligned}$$

Let us rewrite the expression in the first bracket on the right-most side as

$$\left(r^{23}(h; y_{23}) \frac{c}{y_1 - y_2} P^{12} - \frac{c}{y_1 - y_2} P^{12}\, r^{13}(h; y_{13}) \right) + \\ + r^{23}(h; y_{23})\, \tilde{r}^{12}(u; y_{12}) - \tilde{r}^{12}(u; y_{12})\, r^{13}(h; y_{13}).$$

Using fact 4.1, we know that $P^{12}\, r^{13}(h; y_{13}) = r^{23}(h; y_{13})\, P^{12}$, hence the right-most side of (13) equals

$$\frac{r^{23}(h; y_{23}) - r^{23}(h; y_{13})}{y_1 - y_2}\, cP^{12} + r^{23}(h; y_{23})\, \tilde{r}^{12}(u; y_{12}) - \\ - \tilde{r}^{12}(u; y_{12})\, r^{13}(h; y_{13}) + \left[\tilde{r}^{12}(u; y_{12}) - \tilde{r}^{12}(u - h; y_{12})\right]\; r^{13}(h; y_{13}).$$

Passing to the limit $y_2 \to y_1$, we see that

$$\begin{aligned} r^{13}(u; y_{13})\, r^{23}(-u + h; y_{13}) &= -\frac{\partial r^{23}}{\partial y_1}(h; y_{13})\; cP^{12} + r^{23}(h; y_{13})\, \tilde{r}^{12}(u; y_{11}) - \\ &\quad - \tilde{r}^{12}(u; y_{11})\, r^{13}(h; y_{13}) + [\tilde{r}^{12}(u; y_{11}) - \tilde{r}^{12}(u - h; y_{11})]\; r^{13}(h; y_{13}). \end{aligned} \tag{14}$$

We want to apply the operator $\mu\otimes\mathrm{id} : A\otimes A\otimes A \to A\otimes A$ to this equation, where μ is the product in A. Observe that

$$(\mu\otimes\mathrm{id})\left(a^{13}b^{23}\right) = ab,\ (\mu\otimes\mathrm{id})\left(a^{23}b^{12} - b^{12}a^{13}\right) = 0$$

where $a, b \in A\otimes A$ and the notation is best explained by the example $a^{13} = a_1\otimes\mathbb{1}\otimes a_2$ for $a = a_1\otimes a_2$. Moreover, using that $\sum_{i,j} e_{ij}ae_{ji} = \mathrm{tr}(a)\mathbb{1}$ for any $a\in A$ clearly, we derive that for $\mathrm{tr}_1 = \mathrm{tr}\otimes\mathrm{id} : A\otimes A \to A$ we have

$$(\mu\otimes\mathrm{id})\left(a^{23}P^{12}\right) = \mathbb{1}\otimes\mathrm{tr}_1(a).$$

Hence applying $\mu\otimes\mathrm{id}$ to (14) yields

$$r(u; y_{13})\, r(-u+h; y_{13}) = -c\cdot\ \mathbb{1}\otimes\mathrm{tr}_1\left(\frac{\partial r}{\partial y_1}(h; y_{13})\right) +$$

$$+ (\mu\otimes\mathrm{id})\left(\left[\tilde{r}^{12}(u; y_{11}) - \tilde{r}^{12}(u-h; y_{11})\right]\ r^{13}(h; y_{13})\right).$$

Now, take the limit $h\to 0$. The left-hand side of yields $s(u; y_1, y_3)$. As for the right-hand side, we invoke our assumption on the existence of a certain Laurent expansion (2) to derive that

$$\lim_{h\to 0}\frac{\partial r}{\partial y_1}(h; y_{13}) = \frac{\partial r_0}{\partial y_1}(y_1, y_3).$$

Moreover

$$\lim_{h\to 0}\left(\left[\tilde{r}^{12}(u; y_{11}) - \tilde{r}^{12}(u-h; y_{11})\right]\ r^{13}(h; y_{13})\right) = \frac{\partial\tilde{r}^{12}}{\partial u}(u; y_{11})\left(\lim_{h\to 0} r^{13}(h; y_{13})\cdot h\right).$$

Again using (2), we see that the second factor of this last term is simply $\mathbb{1}\otimes\mathbb{1}\otimes\mathbb{1}$. Putting all this together, we end up with

$$s(u; y_1, y_3) = -c\cdot\ \mathbb{1}\otimes\mathrm{tr}_1\left(\frac{\partial r_0}{\partial y_1}(y_1, y_3)\right) + \mu\left(\frac{\partial\tilde{r}}{\partial u}(u; y_1, y_1)\right)\otimes\mathbb{1}.$$

Hence we may write $s(u; y_1, y_2) = \mathbb{1}\otimes\beta(y_1, y_2) + \gamma(u, y_1)\otimes\mathbb{1}$. Note that $\beta(y_1, y_2) = \mathrm{pr}\left(\beta(y_1, y_2)\right) + \frac{\mathrm{tr}\left(\beta(y_1, y_2)\right)}{n}\mathbb{1}$. Using the same trick for $\gamma(u, y_1)$, we may actually write

$$s(u; y_1, y_2) = a(u, y_1)\otimes\mathbb{1} + \mathbb{1}\otimes b(y_1, y_2) + \left(f(u, y_1) + g(y_1, y_2)\right)\mathbb{1}\otimes\mathbb{1}$$

where now both

$$a(u, y_1) = \mathrm{pr}\,\mu\left(\frac{\partial\tilde{r}}{\partial u}(u; y_1, y_1)\right),\ b(y_1, y_2) = -c\cdot\mathrm{pr}\ \mathrm{tr}_1\left(\frac{\partial r_0}{\partial y_1}(y_1, y_2)\right)$$

map to $\mathfrak{sl}_n(\mathbb{C})$. Note that unitarity of $r(u; y_1, y_2)$ implies that $s^{21}(-u; y_2, y_1) = s^{12}(u; y_1, y_2)$. Applying $\mathrm{pr}\otimes\mathbb{1}$ to this equation yields $a(u, y_1) = b(y_2, y_1)$. It follows that both a and b depend on the second variable only and actually coincide, hence

$$s(u; y_1, y_2) = a(y_1)\otimes\mathbb{1} + \mathbb{1}\otimes a(y_2) + \left(f(u, y_1) + g(y_1, y_2)\right)\mathbb{1}\otimes\mathbb{1}.$$

In order to show the statement concerning the form of s, we have to prove that $a(y_1)$ is constant. To this end, we substitute the form of s just calculated into the second equation of the equality stated in lemma 5.2. We derive that

$$\begin{aligned}[]
[a^1(y_1)+a^3(y_3), r^{13}(u_{13};y_1,y_3)] &= r^{13}(u_{13};y_1,y_3)\cdot \\
\cdot\left(f(u_{32},y_1)+f(u_{21},y_2)-f(u_{12},y_1)-f(u_{23},y_2)\right). &
\end{aligned} \tag{15}$$

Let us focus on the left-hand side. This equals

$$\left[a^1(y_1)+a^3(y_3),\frac{cP^{13}}{y_1-y_3}\right]+\left[a^1(y_1)+a^3(y_3),\tilde{r}^{13}(u_{13};y_1,y_3)\right].$$

By fact 4.1, we may rewrite the first summand as

$$\frac{a^1(y_1)-a^1(y_3)}{y_1-y_3}cP^{13}+\frac{a^3(y_3)-a^3(y_1)}{y_1-y_3}cP^{13}$$

thus the limit $y_3\to y_1$ of the left-hand side of (15) is given by

$$\frac{d}{dy_1}\left(a^1(y_1)-a^3(y_1)\right)cP^{13}+\left[a^1(y_1)+a^3(y_1),\tilde{r}^{13}(u_{13};y_1,y_1)\right].$$

In particular, the limit $y_3\to y_1$ of the right-hand side of (15) exists as well. But $r^{13}(u_{13};y_1,y_3)$ has a pole along $y_1=y_3$ and the other factor on the right-hand side is independent of y_3. Hence we conclude that

$$f(u_{32},y_1)+f(u_{21},y_2)-f(u_{12},y_1)-f(u_{23},y_2)=0.$$

In particular, the left-hand side of (15) equals zero. Focusing on the polar part yields

$$\left[a^1(y_1)+a^3(y_3),\frac{cP^{13}}{y_1-y_3}\right]=0$$

which, by the above, implies

$$\frac{d}{dy_1}\left(a^1(y_1)-a^3(y_1)\right)cP^{13}=0.$$

But then $\frac{da}{dy}(y)$ must be zero, so that $a(y)=a\in\mathfrak{sl}_n(\mathbb{C})$ is constant. Therefore $s(u;y_1,y_2)=a\otimes\mathbb{1}+\mathbb{1}\otimes a+(f(u,y_1)+g(y_1,y_2))\,\mathbb{1}\otimes\mathbb{1}$. By (15) we also see that a is an infinitesimal symmetry of $r(u;y_1,y_2)$.

Finally, we want to prove the statement concerning $r_0(y_1,y_2)$. Clearly, we may write

$$r_0(y_1,y_2)=\overline{r}_0(y_1,y_2)+\alpha(y_2,y_1)\otimes\mathbb{1}-\mathbb{1}\otimes\alpha(y_1,y_2)+h(y_1,y_2)\mathbb{1}\otimes\mathbb{1}$$

with α mapping to $\mathfrak{sl}_n(\mathbb{C})$. Note that in the discussion above we derived that

$$a=b(y_1,y_2)=-c\cdot\mathrm{pr}\ \mathrm{tr}_1\left(\frac{\partial r_0}{\partial y_1}(y_1,y_2)\right).$$

Since both $\overline{r}_0$ and α map to $\mathfrak{sl}_n(\mathbb{C})$, so will their partial derivatives. This implies

$$a = -cn \cdot \mathrm{pr}\left(-\left(\frac{\partial}{\partial y_1}\right)\alpha(y_1,y_2) + \frac{\partial h}{\partial y_1}(y_1,y_2)\cdot \mathbb{1}\right).$$

Hence $\frac{\partial}{\partial y_1}\alpha(y_1,y_2) = \frac{a}{cn}$, which gives the formula for α. In particular, $\alpha(y_1,y_2)$ does not depend on the second argument. This completes the proof of the formula for r_0. □

Before we finally prove theorem 5.1, we need to state one more easy fact:

Fact 5.4. a) For any $x, y, \phi \in A$, $i \in \{1,2\}$ and $\phi^1 = \phi \otimes \mathbb{1}$ respectively $\phi^2 = \mathbb{1} \otimes \phi$, we have

$$(\mathrm{pr} \otimes \mathrm{pr})\left[x \otimes y, \phi^i\right] = \left[(\mathrm{pr}\otimes\mathrm{pr})\left(x\otimes y\right), \phi^i\right].$$

b) Let r be a non-degenerate solution of (1) and $[r, 1 \otimes a] = 0$ for some $a \in \mathfrak{sl}_n(\mathbb{C})$. Then $a = 0$.

Proof. a) is straightforward. As to b), write $r = \sum_{i\in I} r_i' \otimes r_i''$ for some index set I and let $\varphi : A \otimes A \to \mathrm{End}(A)$ denote the isomorphism given by $X \otimes Y \mapsto (Z \mapsto X\,\mathrm{tr}\,(YZ))$. Then

$$0 = \varphi\left([r, 1\otimes a]\right)(b) = \sum_{i\in I} r_i'\,\mathrm{tr}\left([r_i'', a]\, b\right) = \sum_{i\in I} r_i'\,\mathrm{tr}\left(r_i''\,[a,b]\right) = \varphi(r)\left([a,b]\right)$$

for all $b \in A$. Now r is non-degenerate, hence $\varphi(r)$ is an isomorphism. This yields $[a,b] = 0$ for all $b \in A$, especially all $b \in \mathfrak{sl}_n(\mathbb{C})$. But the Lie bracket is non-degenerate on $\mathfrak{sl}_n(\mathbb{C})$, hence $a = 0$. □

Proof of Theorem 5.1.
(1) By lemma 2.5 $\bar{r}_0$ is a unitary solution of the CYBE (3). The rest is immediate by lemma 4.3 and the fact that $(\mathrm{pr}\otimes\mathrm{pr})\,(P)$ is the Casimir element of $\mathfrak{sl}_n(\mathbb{C}) \otimes \mathfrak{sl}_n(\mathbb{C})$ with respect to the trace form $(x,y) \mapsto \mathrm{tr}(x\cdot y)$.

(2) Setting $u_1 = u$, $u_2 = 0$ and $u_3 = -u$ in lemma 5.2, we derive that $r(u; y_1, y_2)$ satisfies the QYBE (4) for u fixed if and only if

$$s^{23}(u; y_2, y_3)\, r^{13}(2u; y_1, y_3) = r^{13}(2u; y_1, y_3)\, s^{23}(-u; y_2, y_3).$$

Applying lemma 5.3 this is equivalent to

$$[r(u; y_1, y_2), 1 \otimes a] = 0 \tag{16}$$

which, by fact 5.4 b) is equivalent to $a = 0$. By lemma 5.3 this last condition holds if and only if either of conditions (b) or (c) of the theorem are satisfied. It remains to show equivalence with condition (d). To this end, recall (7). Denote the right-hand side of this equation by $AYBE[r_0](y_1, y_2, y_3)$. Then (d) simply reads

$(\mathrm{pr} \otimes \mathrm{pr} \otimes \mathrm{pr})\, AYBE[\overline{r}_0](y_1, y_2, y_3) = 0$. To show the equivalence of this with $a = 0$, express r_0 in terms of $\overline{r}_0$ as stated in lemma 5.3. Then

$$-cn \cdot (\mathrm{pr} \otimes \mathrm{pr} \otimes \mathrm{pr})\, (AYBE\, [r_0]\, (y_1, y_2, y_3) - AYBE\, [\overline{r}_0]\, (y_1, y_2, y_3)) =$$
$$= (y_1 - y_3)\, \overline{r}_0^{13}(y_1, y_3)\, a^2 + (y_1 - y_2)\, \overline{r}_0^{12}(y_1, y_2)\, a^3 + (y_3 - y_2)\, \overline{r}_0^{23}(y_2, y_3)\, a^1.$$

It is immediate by (7) that $(\mathrm{pr} \otimes \mathrm{pr} \otimes \mathrm{pr})\, AYBE\, [r_0]\, (y_1, y_2, y_3) = 0$. Hence if a is zero, this implies $(\mathrm{pr} \otimes \mathrm{pr} \otimes \mathrm{pr})\, AYBE[\overline{r}_0] = 0$. On the other hand, assuming $(\mathrm{pr} \otimes \mathrm{pr} \otimes \mathrm{pr})\, AYBE[\overline{r}_0] = 0$ we deduce

$$(y_1 - y_3)\, \overline{r}_0^{13}(y_1, y_3)\, a^2 + (y_1 - y_2)\, \overline{r}_0^{12}(y_1, y_2)\, a^3 + (y_3 - y_2)\, \overline{r}_0^{23}(y_2, y_3)\, a^1 = 0.$$

We will show that this implies $a = 0$. Indeed, by lemma 4.3 we know that $r_0(y_1, y_2) = \frac{cP}{y_1 - y_2} + \tilde{r}_0(y_1, y_2)$ with $\tilde{r}_0$ being defined along $y_1 = y_2$ and similarly for $\overline{r}_0$. Hence passing to the limit $y_1, y_2, y_3 \to y$ yields

$$(\mathrm{pr} \otimes \mathrm{pr} \otimes \mathrm{pr})\, \left[P^{13} a^2 + P^{12} a^3 + P^{23} a^1\right] = 0.$$

Let us write $a = \sum a_{ij} e_{ij}$. Looking at the coefficient of $e_{ij} \otimes e_{ji} \otimes e_{ij}$ in the above equation for $i \neq j$, we derive $a_{ij} = 0$. But then projecting the above equation to $e_{12} \otimes e_{21} \otimes \mathfrak{sl}_n(\mathbb{C})$, we may deduce that $a = 0$.

(3) As we just saw, the conditions of (2) are satisfied if $a = 0$. But a is an infinitesimal symmetry of r by lemma 5.3, hence one of r_0. Invoking fact 5.4 a), we deduce that a is an infinitesimal symmetry of $\overline{r}_0$ and so $a = 0$. □

6. Uniqueness of lifts from CYBE to AYBE

In this section we will prove part ii) of theorem 1.1:

Theorem 6.1. [3, Theorem 6] *Let $r(u; y_1, y_2)$ and $s(u; y_1, y_2)$ be a unitary solutions of (1) of the form (2). Assume that the corresponding solution $\overline{r}_0(y_1, y_2) = (\mathrm{pr} \otimes \mathrm{pr})\, (r_0(y_1, y_2))$ of the CYBE (3) is non-degenerate, has no infinitesimal symmetries and that $\overline{s}_0(y_1 y_2) = \overline{r}_0(y_1, y_2)$. Then there exists a meromorphic function $g : \mathbb{C} \to \mathbb{C}$ such that $s(u; y_1, y_2) = \exp\big(u\big(g(y_2) - g(y_1)\big)\big) r(u; y_1, y_2)$.*

Proof. First, we show that r is uniquely determined by r_0. By lemma 3.5 r is uniquely determined by r_0 and r_1 and moreover r_1 is a solution of a certain equation in r_0 which is given by (7). If $r_1' \neq r_1$ was a solution of (7) with the same properties as r_1, then taking the difference we would obtain a meromorphic function $\alpha : (\mathbb{C}^2, 0) \to A \otimes A$ with $\alpha^{21}(y_2, y_1) = \alpha(y_1, y_2)$ and

$$\alpha^{12}(y_1, y_2) + \alpha^{13}(y_1, y_3) + \alpha^{23}(y_2, y_3) = 0. \tag{17}$$

Using lemma 4.2, we also know that the residue of $r(u; y_1, y_2)$ near $y_1 = y_2$ is independent of u. Comparing this to the Laurent expansion (2), we derive that $r_1(y_1, y_2)$ has no poles along $y_1 = y_2$, hence the same is true for $\alpha(y_1, y_2)$. To prove that r_1 is determined by r_0, we need only show that α is already zero. Choosing $y_3 = y_2$

and then applying $\mathrm{pr} \otimes \mathrm{id} \otimes \mathrm{id}$ to (17) we derive that $(\mathrm{pr} \otimes \mathrm{id})\left(\alpha(y_1, y_2)\right) = 0$. Similarly, $(\mathrm{id} \otimes \mathrm{pr})\left(\alpha(y_1, y_2)\right) = 0$, hence $\alpha(y_1, y_2) = f(y_1, y_2)\, \mathbb{1} \otimes \mathbb{1}$ where f is a meromorphic function such that $f(y_1, y_2) + f(y_1, y_3) + f(y_2, y_3) = 0$. Since $r_1(y_1, y_2)$ has no pole along $y_1 = y_2$ by lemma 4.2, $\alpha(y_1, y_1)$ exists. We may deduce that $2f(y_1, y_2) = -f(y_2, y_2)$, so f depends only on the second variable. But then choosing $y_2 = y_1 = y_3$ we read $3f(y_1, y_1) = 0$, thus $f = 0$.We have proved that r is uniquely determined by r_0.

It remains to prove that, provided $\overline{r}_0$ has no infinitesimal symmetries, r can be uniquely recovered from $\overline{r}_0$ up to the factor $\exp\left(u\left(g(y_2) - g(y_1)\right)\right)$ for some meromorphic function $g : \mathbb{C} \to \mathbb{C}$. Note that this is equivalent to showing that $r_0(y_1, y_2)$ is uniquely determined by $\overline{r}_0(y_1, y_2) = (\mathrm{pr} \otimes \mathrm{pr})\left(r_0(y_1, y_2)\right)$ up to a summand of the form $\left(g(y_2) - g(y_1)\right) \mathbb{1} \otimes \mathbb{1}$. By assumption $\left(s_0(y_1, y_2), s_1(y_1, y_2)\right)$ is another tuple satisfying (7) such that

$$s_0^{21}(y_2, y_1) = -s_0(y_1, y_2),\ s_1^{21}(y_2, y_1) = s_1(y_1, y_2).$$

We claim that

$$s_0(y_1, y_2) = r_0(y_1, y_2) + \left(g(y_2) - g(y_1)\right) \mathbb{1} \otimes \mathbb{1}.$$

Since $\overline{s}_0(y_1, y_2) = \overline{r}_0(y_1, y_2)$, we may write

$$s_0(y_1, y_2) = r_0(y_1, y_2) + \phi^1(y_1, y_2) - \phi^2(y_2, y_1) + \psi(y_1, y_2)\, \mathbb{1} \otimes \mathbb{1}$$

for a $\mathfrak{sl}_n(\mathbb{C})$ valued function ϕ and a scalar function ψ. Denoting the left-hand side of (7) by $LHS(r)$, we have

$$0 = (\mathrm{pr} \otimes \mathrm{pr} \otimes \mathrm{pr})\left(LHS(s) - LHS(r)\right) = \overline{r}_0^{12}(y_1, y_2)\ \left[\phi^3(y_3, y_2) - \phi^3(y_3, y_1)\right] +$$
$$+ \overline{r}_0^{23}(y_2, y_3)\ \left[\phi^1(y_1, y_3) - \phi^1(y_1, y_2)\right] + \overline{r}_0^{13}(y_1, y_3)\ \left[\phi^2(y_2, y_3) - \phi^2(y_2, y_1)\right].$$

If the function ϕ is not constant then contracting this equation with a generic functional in the third component we derive that $\overline{r}_0$ is a sum of two decomposable tensors, that is $\bar{r}_0 = a_1 \otimes b_1 + a_2 \otimes b_2$ where all terms depend on y_1, y_2. But $\bar{r}_0$ is nondegenerate by assumption, so $\mathrm{span}_{\mathbb{C}}\left(\{a_1, a_2\}\right) \cong \mathfrak{sl}_n(\mathbb{C})$, which is impossible for any $n \geq 2$. Thus $\phi \in \mathfrak{sl}_n(\mathbb{C})$ is constant. Applying $(\mathrm{pr} \otimes \mathrm{pr} \otimes \mathrm{id})$ to $LHS(s) - LHS(r)$ yields

$$\begin{aligned}(\mathrm{pr} \otimes \mathrm{pr} \otimes \mathrm{id})\left(s_1^{12}(y_1, y_2) - r_1^{12}(y_1, y_2)\right) = (\mathrm{pr} \otimes \mathrm{pr} \otimes \mathrm{id})\big(r_0^{12}(y_1, y_2)\phi^1 - \\ - \phi^2 r_0^{12}(y_1, y_2)\big) - \phi^1\phi^2 + \left(\psi(y_1, y_3) - \psi(y_2, y_3)\right) \overline{r}^{12}(y_1, y_2).\end{aligned} \tag{18}$$

This implies that $\psi(y_1, y_3) - \psi(y_2, y_3)$ is actually independent of y_3, hence equal to some function $\beta(y_1, y_2)$. Also, we know by unitarity of r that $\psi(y_1, y_2) = -\psi(y_2, y_1)$, thus $\beta(y_1, y_2) = \psi(y_1, y_3) + \psi(y_3, y_2)$. It follows from lemma 4.2 that r_0 and s_0 have the same pole along $y_1 = y_2$, hence $\psi(y_1, y_1)$ exists and we may deduce that $\beta(y_1, y_2) = \psi(y_1, y_1) + \psi(y_1, y_2) = \psi(y_1, y_2)$. Thus the definition of β reads

$\psi(y_1, y_2) = \psi(y_1, y_3) - \psi(y_2, y_3)$. Therefore, defining $g(y) = \psi(y, a)$ for some fixed $a \in \mathbb{C}$, we have $\psi(y_1, y_2) = g(y_1) - g(y_2)$. Altogether

$$s_0(y_1, y_2) = r_0(y_1, y_2) + \phi^1 - \phi^2 + (g(y_1) - g(y_2))\, \mathbb{1} \otimes \mathbb{1} \tag{19}$$

Since s_0 and r_0 are both meromorphic, so is ψ and thus also g.

Next, we replace $r(u; y_1, y_2)$ by $\exp\left(u\left(g(y_2) - g(y_1)\right)\right) r(u; y_1, y_2)$ and hence may assume that $g = 0$ in the above formula for s_0. Thus (18) yields

$$(\mathrm{pr} \otimes \mathrm{pr})\left(s_1(y_1, y_2) - r_1(y_1, y_2)\right) = (\mathrm{pr} \otimes \mathrm{pr})\left(r_0(y_1, y_2)\phi^1 - \phi^2 r_0(y_1, y_2)\right) - \phi^1\phi^2.$$

We exchange the first two components, make the substitutions $y_1 \leftrightarrow y_2, y_2 \leftrightarrow y_1$ and use unitarity of r for both sides of the resulting equation. Comparing the result with the above equation, we derive

$$(\mathrm{pr} \otimes \mathrm{pr})\left(r_0(y_1, y_2)\phi^1 - \phi^2 r_0(y_1, y_2)\right) = (\mathrm{pr} \otimes \mathrm{pr})\left(-r_0(y_1, y_2)\phi^2 + \phi^1 r_0(y_1, y_2)\right).$$

By fact 5.4 a) we deduce that $[\overline{r}_0(y_1, y_1), \phi^1 + \phi^2] = 0$. But then ϕ is an infinitesimal symmetry of $\overline{r}_0$, so $\phi = 0$. Thus $s_0 = r_0$. ☐

Remark 6.2. *By theorem 5.1 (1), the assumption of theorem 6.1 on the non-degeneracy of $\bar{r}_0$ is automatically satisfied if r itself is non-degenerate. Moreover, we deduce from the proof of theorem 6.1 that in that case r is already uniquely determined by r_0.*

Corollary 6.3. *In the notations of theorem 6.1, assume that $r_0(y_1, y_2)$ and $s_0(y_1, y_2)$ have the same poles on $\left(\mathbb{C}^2 \setminus V\left((y_1 - y_2)\right)\right)$. Then g is a holomorphic function. Thus, r and s are gauge equivalent.*

Proof. It follows from the assumption and lemma 4.2 that the poles of r_0 and s_0 coincide. By (19) this implies that g is holomorphic. The remaining statement follows from the discussion in example 3.2. ☐

Combining the above corollary with theorem 2.3 yields the final result of this section, finishing the proof of theorem 1.1 ii).

Corollary 6.4. *In the notations of theorem 6.1, assume that both r and s can be obtained by the procedure described in section 2. Then r and s are gauge equivalent.*

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